# Möbius transformations

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# 1. Introduction

# Definition

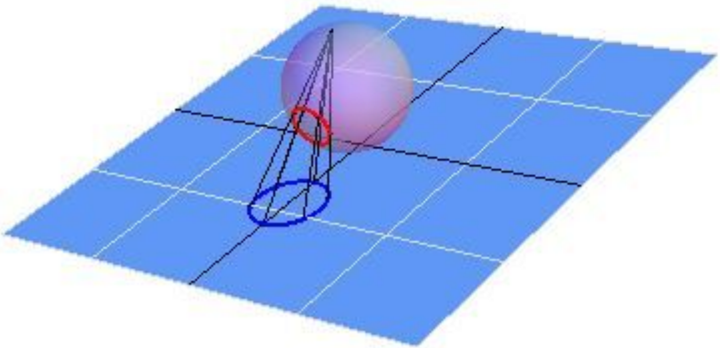
A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism if $f$ is a bijection and if both $f$ and $f^{-1}$ are continuous.

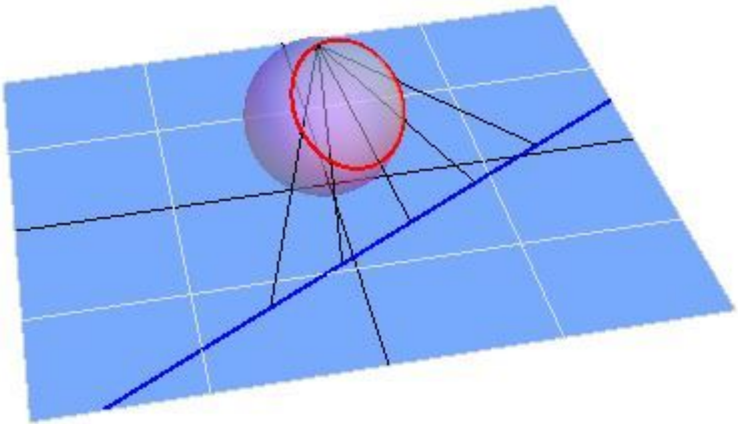
Note that,

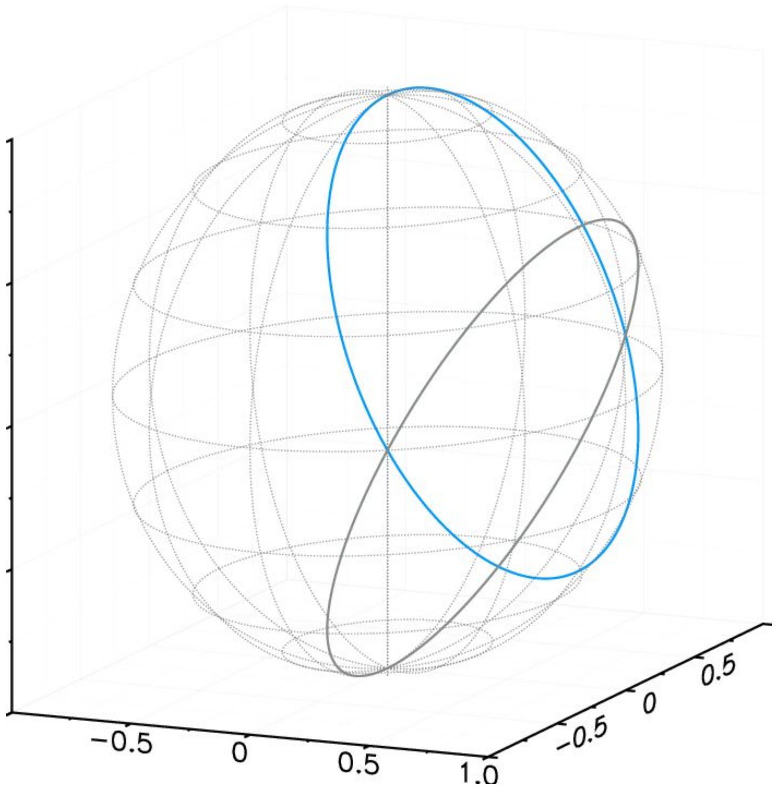
$$Homeo(\overline{\mathbb{C}}) = \{f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} : f \text{ is homeomorphisms }\}$$

Let $Homeo^{C}(\overline{\mathbb{C}})$ be a subset of the group $Homeo(\overline{\mathbb{C}})$ that contains all those homeomorphisms of $\overline{\mathbb{C}}$ taking the circle in $\overline{\mathbb{C}}$ to circle in $\overline{\mathbb{C}}$

# Recall Circle in Riemann Sphere








Circle in Riemann sphere

# Example

1. There are some elements that in $Homeo(\overline{\mathbb{C}})$ but not in $Homeo^C(\overline{\mathbb{C}})$

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$f(z) = \begin{cases} z & Re(z) < 0 \\ z + iRe(z) & Re(z) \geq 0 \\ \infty & z = \infty \end{cases}$$

$$f^{-1}(z) = \begin{cases} z & Re(z) < 0 \\ z - iRe(z) & Re(z) \geq 0 \\ \infty & z = \infty \end{cases}$$

Note that, $f$ is bijective and both $f$ and $f^{-1}$ are continuous. Therefore, $f \in Homeo(\overline{\mathbb{C}})$ . However, since the image of $\overline{\mathbb{R}}$ under $f$ is not a circle in $\overline{\mathbb{C}}$ . Therefore, it is not in $Homeo^C(\overline{\mathbb{C}})$

# Example

2. The element $f$ of $Homeo(\overline{\mathbb{C}})$ is defined by

$$f(z) = az + b \quad \text{for} \quad z \in \mathbb{C} \text{ and } f(\infty) = \infty$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, and it is an element in $Homeo^C(\overline{\mathbb{C}})$

Proof

Recall that the equation of the circle in $\mathbb{C}$ is in the form,

$$\alpha z\bar{z} + \beta z + \overline{\beta}\bar{z} + \gamma = 0$$
$$\text{where } \alpha, \gamma \in \mathbb{R} \text{ and } \beta \in \mathbb{C} \text{ and where } \alpha \neq 0 \qquad \text{....(1)}$$

Then every Euclidean line in $\mathbb{C}$ can be explained in the form,

$$\beta z + \overline{\beta}\bar{z} + \gamma = 0 \qquad \text{....(2)}$$
$$\text{where } \beta \in \mathbb{C} \text{ and } \gamma \in \mathbb{R}$$

Now, we need to show f satisfies these equations

# Proof

Let $w = az + b$, then $z = \dfrac{w-b}{a}$

Put $z = \dfrac{w-b}{a}$ into the equation (2)

We have,

$$\beta z + \overline{\beta}\overline{z} + \gamma = \frac{\beta(w-b)}{a} + \overline{\frac{\beta(w-b)}{a}} + \gamma$$

$$= \frac{\beta}{a}w + \overline{\frac{\beta}{a}}\overline{w} - \frac{\beta}{a}b - \overline{\frac{\beta}{a}}b + \gamma$$

$$= 0$$

This shows that w also satisfies the equation of a Euclidean line in $\mathbb{C}$

# Continue

Let $w = az + b$, then $z = \dfrac{w-b}{a}$

Put $z = \dfrac{w-b}{a}$ into the equation (1)

$$\begin{aligned}
\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma &= \alpha\frac{1}{a}(w-b)\overline{\frac{1}{a}(w-b)} + \beta\frac{1}{a}(w-b) + \overline{\beta}\overline{\frac{1}{a}(w-b)} + \gamma \\
&= \frac{\alpha}{|a|^2}(w-b)\overline{w-b} + \frac{\alpha}{a}\beta\alpha(w-b) + \frac{\overline{\alpha}}{a}\overline{\beta}\overline{\alpha}(\overline{w-b}) + \frac{\alpha}{|a|^2}\frac{|\beta|^2 a}{\alpha} - \frac{\alpha}{|a|^2}\frac{|\beta|^2 a}{\alpha} + \gamma \\
&= \frac{\alpha}{|a|^2}((w-b+\frac{\overline{\beta}a}{\alpha})(\overline{w-b}+\frac{\beta\overline{a}}{\overline{\alpha}}) - \frac{|b|^2}{\overline{a}} + \gamma \\
&= \frac{\alpha}{|a|^2}\left|w-b+\frac{\overline{\beta}a}{\alpha}\right|^2 - \frac{|b|^2}{\overline{a}} + \gamma = 0
\end{aligned}$$

Then it satisfy the equation of the circle in $\mathbb{C}$ .

# Example

3. The element $J$ of $Homeo(\overline{\mathbb{C}})$ defined by $J(z) = \frac{1}{z}$ for $z \in \mathbb{C} - \{0\}$, $J(0) = \infty$, and $J(\infty) = 0$

and it is an element in $Homeo^{C}(\overline{\mathbb{C}})$

## Proof

Let $w = \frac{1}{z}$ , then $z = \frac{1}{w}$ ,put it into equation (1)

We have, $\alpha \frac{1}{w}\overline{\frac{1}{w}} + \beta \frac{1}{w} + \overline{\beta}\overline{\frac{1}{w}} + \gamma = 0$

Multiply both sides by $w\overline{w}$

Then we have,

$$\alpha + \beta\overline{w} + \overline{\beta}w + \gamma w\overline{w} = 0$$

Then it satisfy the equation of the circle in $\mathbb{C}$ .

# Definiton

A Möbius transformation is a function $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$M(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex constants and *ad-bc* ≠ 0.

# Some remarks for $M(\infty)$ and $M(0)$

For $M(\infty)$,

$$M(\infty) = \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}$$

Since a or c cannot be both zero by the assumption *ad-bc* ≠ 0, then it is well defined.
Also, it equals to $\infty$ if and only if c=0.

For *M*(0),

$$M(0) = \frac{b}{d}$$

Then we have M(0)=0 if and only if b=0

# Theorm

Consider the Möbius transformations,

If c=0, $M(z) = \frac{a}{d}z + \frac{b}{d}$

If $c \neq 0, M(z) = f(J(g(z)))$, where $g(z) = c^2 z + cd$ and $f(z) = -(ad - bc)z + \frac{a}{c}$

## Proof

For c = 0, it is a direct computation.

For $c \neq 0$, $M(z) = \frac{az+b}{cz+d} = \frac{acz+bc}{c^2z+dc} = \frac{acz+ad-ad+bc}{c^2z+dc} = \frac{a}{c} - \frac{ad-bc}{c^2z+dc}$

Then we have $g(z) = c^2 z + cd$ and $f(z) = -(ad - bc)z + \frac{a}{c}$

Note that, $J(z) = \frac{1}{z}$ for $z \in \mathbb{C} - \{0\}, J(0) = \infty,$ and $J(\infty) = 0$

Therefore, $M(z) = f(J(g(z)))$

# By previous example

The element $f$ of $Homeo(\overline{\mathbb{C}})$ is defined by

$$f(z) = az + b \quad \text{for} \quad z \in \mathbb{C} \text{ and } f(\infty) = \infty$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, and it is an element in $Homeo^C(\overline{\mathbb{C}})$

The element $J$ of $Homeo(\overline{\mathbb{C}})$ defined by $J(z) = \frac{1}{z}$ for $z \in \mathbb{C} - \{0\}$, $J(0) = \infty$, and $J(\infty) = 0$

and it is an element in $Homeo^C(\overline{\mathbb{C}})$

# Corollary

1. Möb$^+$ $\subset Homeo(\overline{\mathbb{C}})$

Since by the previous theorem, we know that Möbius transformations is a composition of homeomorphisms, therefore it is a subset of $Homeo(\overline{\mathbb{C}})$

2. Möb$^+$ $\subset Homeo^C(\overline{\mathbb{C}})$

Since Möbius transformations is a composition of functions which have a property that take circle in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$

# Example when ad-bc = 0

Let $p : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$p = \frac{az+b}{cz+d}, \text{ where a,b,c,d} \in \mathbb{C} \text{ and } ad-bc=0$$

$Then$ p is not a homeomorphism of $\overline{\mathbb{C}}$

Proof

We have $ad - bc = 0$, then$ad = bc$,

$$p = \frac{az+b}{cz+d} = \frac{a^2z+ba}{acz+da} = \frac{a^2z+ba}{acz+bc} = \frac{a}{c}$$

Therefore it is a constant function

# Theorem

Let M(z) be a Möbius transformation and $M(0, 1, \infty) = (0, 1, \infty)$

Then M is an identiy transformation. M(z) = z for any z in $\overline{\mathbb{C}}$

Proof

$$\text{We have } M(z) = \frac{az+b}{cz+d}$$

$$For z = 0, M(0) = 0 \Leftrightarrow b = 0$$

$$For z = 1, M(1) = 1 \Leftrightarrow a = d$$

$$For z = \infty, M(\infty) = \infty \Leftrightarrow c = 0$$

$$\text{Then, } M(z) = \frac{az}{a} = z$$

# 2. Transitivity Properties

# Properties

Möb$^+$ acts uniquely triply transitivty on $\overline{\mathbb{C}}$

Proof

First, prove the uniqueness

Let $(z_1, z_2, z_3), (w_1, w_2, w_3)$ be distinct point in $\overline{\mathbb{C}}$.

Let n, m are elements in Möb$^+$ such that

$$n(z_1) = w_1 = m(z_1), n(z_2) = w_2 = m(z_2), n(z_3) = w_3 = m(z_3)$$

Then $m^{-1} \circ n$ is an identity

By the previous theorem, we know that is identity.

Then m = n

# Continue

Now, prove the existence.

Let $(z_1, z_2, z_3)$ be distinct point in $\overline{\mathbb{C}}$.

Now, we need to construct a Möbius transformation m such that

$$m(z_1) = 0, m(z_2) = 1, m(z_3) = \infty$$

Then, Let $m = \dfrac{z - z_1}{z - z_3} \dfrac{z_2 - z_3}{z_2 - z_1}$

$$m = \frac{z(z_2 - z_3) + z_1(z_3 - z_2)}{z(z_2 - z_1) + z_3(z_1 - z_2)}$$

$Then\ \ a = z_2 - z_3, b = z_1(z_3 - z_2), c = (z_2 - z_1), d = z_3(z_1 - z_2)$

They are complex constant and $ac - bd \neq 0$

# Definition

A group $G$ acts on a set $X$ if there is a homomorphism from $G$ in to the group bij($X$) of bijections of $X$

## Definition

$G$ acts transitively on $X$ if for each pair $x$ and $y$ of elements of $X$, there exist some element $g$ of $G$ satisfying $g(x) = y$

## Lemma

Suppose a group $G$ acts on a set $X$, and let $x_0$ be a point of $X$. Suppose for each point $y$ of $X$, there exists an element $g$ of G so that $g(y) = x_0$. Then, $G$ acts transitively on a set $X$

# Proof of lemma

Let x, y be the element in X and $g_x, g_y$be the element in G such that

$$g_y(y) = x_0 = g_x(x),$$
$$x = g_x^{-1}(x_0) = g_x^{-1} \circ g_y(y)$$

Since x, y is a pair of element in X and $g_x^{-1} \circ g_y$ is an element in G
Therefore, it is transitively

It also proves that *G* acts uniquely transitively on a set *X*

# Theorem

$\text{Möb}^+$ acts transitively on the set $\mathcal{C}$ on circles in $\overline{\mathbb{C}}$

proof

First, we need to show the fact that any triple of distinct points in $\overline{\mathbb{C}}$ defines a unique circle in $\overline{\mathbb{C}}$

Let $(z_1, z_2, z_3)$ be distinct points of $\overline{\mathbb{C}}$ .
If they are not collinear, then there exist a unique **Euclidean circle** passing through all three points.
If they are collinear, then there exists a unique **Euclidean line** passing through all three.
If one of the $(z_1, z_2, z_3)$ is $\infty$ , then there is a unique **Euclidean line** passing through the other two.

# Continue

Let A, B be circles in $\overline{\mathbb{C}}$

Choose a triple distinct points on A and B respectively.

Let m be the  Möbius transformations taking the triple of distinct points determining A to the

triple of distinct points determining B.

As m(A) and B are two circles in $\overline{\mathbb{C}}$ that pass through the same triple of distinct points, we have that m(A) = B

# Example(show the action is not uniquely transitive)

Let $(z_1, z_2, z_3)$ be a triple of distinct points and let A be the circle in $\overline{\mathbb{C}}$ determined by $(z_1, z_2, z_3)$

Then the identity takes A to A.

The Möbius transformations taking $(z_1, z_2, z_3)$ to $(z_1, z_3, z_2)$ is also takes A to A

# Theorm

Möb$^+$ acts transitively on the set $D$ on discs in $\overline{\mathbb{C}}$

## Proof

First, recall the definition of disc

Define a disc in $\overline{\mathbb{C}}$ to be one of the components of the complement in $\overline{\mathbb{C}}$ of a circle in $\overline{\mathbb{C}}$

Note that every disc in $\overline{\mathbb{C}}$ determines a unique circle in $\overline{\mathbb{C}}$, and that every circle in $\overline{\mathbb{C}}$ determines two disjoint discs in $\overline{\mathbb{C}}$.

Then,

Let A, B be two discs in $\overline{\mathbb{C}}$, where A is determined by circle $C_A$ and B is determined by circle $C_B$

# Continue

Note that, m(A) can either produce B or the other disc determined by $C_B$

Case 1: If m(A)=B, we proved the theorem.

Case2:

Recall $J(z) = \frac{1}{z}, J(0) = \infty, J(\infty) = 0, J(1) = 1$

Then J takes $\overline{\mathbb{R}}$ to itself, so J interchanges two discs determined by $\overline{\mathbb{R}}$

Now, let n be a Möbius transformations such that $n(A) = \overline{\mathbb{R}}$

Then, $n^{-1} \circ J \circ n$ takes A to itself and interchanges two discs determined by A.

# 3. Transformation

# Definition

Two Möbius transformations $m_1, m_2$ are conjugate if there exist some Möbius transformations *p* so that $m_2 = p \circ m_1 \circ p^{-1}$

# Definition

A ***fixed point*** of the Möbius transformations $m$ is a point $z$ of satisfying $m(z) = z$, where m is not an identity

## Theorem

Suppose m and n are Möbius transformations that are conjugate. Then, m and n have the same number of fixed points in $\overline{\mathbb{C}}$

# proof

Since m and n are conjugate ,then by definition. There is a Möbius transformations *p such that*

$$m = p \circ n \circ p^{-1} \text{ and } n = p^{-1} \circ m \circ p$$

If n fixes a point x in $\overline{\mathbb{C}}$, then $m = p \circ n \circ p^{-1}$ fixes p(x)

$$m(p(x)) = p \circ n \circ p^{-1}(p(x)) = p(n(x)) = p(x)$$

If m fixes a point y in $\overline{\mathbb{C}}$, then n fixes $p^{-1}(y)$

$$n(p^{-1}(y)) = p^{-1} \circ m \circ p(p^{-1}(y)) = p^{-1}(m(y)) = p^{-1}(y)$$

Therefore, they have the same number of fixed points

# Conjugate a Möbius transformations into a standard form

Suppose m is not an identity.

Suppose x is the only fixed point of m in $\overline{\mathbb{C}}$

Let y be any point on $\overline{\mathbb{C}}$ but not equal to x.

Then $(x, y, m(y))$ is a triple of distinct points of $\overline{\mathbb{C}}$

Let p be the Möbius transformations taking $(x, y, m(y))$ to $(\infty, 0, 1)$

We have,

$$p \circ m \circ p^{-1}(\infty) = p(m(x)) = p(x) = \infty$$

Since x is a fixed point and $m(x) = x$

# Continue

Since $p \circ m \circ p^{-1}$ only fixed on $\infty$

Then,

Let $p \circ m \circ p^{-1}(z) = az + b$ for some $a \neq 0$

$p \circ m \circ p^{-1}(0) = p(m(y)) = 1$, then $b = 1$

Since there is no solution for $p \circ m \circ p^{-1}(z) = z$ in $\mathbb{C}$
then $a = 1$

We have $n = p \circ m \circ p^{-1}(z) = z + 1$ and it is the standard form of n

# Example

Find the Möbius transformation p conjugating m to its standard form when $m(z) = \frac{z}{z+1}$

## Answer

Let $m(z) = \frac{z}{z+1}$

First, we need to find the fixed point.

Let $m(z) = z$,

$\frac{z}{z+1} = z$

$z^2 + z = z$

Then, the only fixed point of m is 0

Now, choose some point in $\overline{\mathbb{C}}$ but not equal to 0

Since we have $m(\infty) = 1$,

We can take p to be the Möbius transformation from the triple of $(0, \infty, 1)$ to the triple of $(\infty, 0, 1)$

Then we have $p(z) = \frac{1}{z}$

# Case on two fixed points

Suppose x and y are two fixed points of m in $\overline{\mathbb{C}}$

Let q be a Möbius transformation such that $q(x) = 0 \text{ and } q(y) = \infty$

By definition, $q \circ m \circ q^{-1}(\infty) = q(m(y)) = q(y) = \infty$

$$q \circ m \circ q^{-1}(0) = q(m(x)) = q(x) = 0$$

Then we may write,

$$q \circ m \circ q^{-1}(z) = az, \text{for some elements in } \mathbb{C} \text{ but not equal to } 0 \text{ or } 1$$

And a is called as the multiplier of m

# Example

Find the Möbius transformation q conjugating m to its standard form and multipier of m when $m(z) = \frac{2z+1}{z+1}$

## Answer

$$\text{Let } m(z) = \frac{2z+1}{z+1}$$

First, find the fixed points.

$$m(z) = \frac{2z+1}{z+1} = z$$

$$z^2 - z - 1 = 0$$

$$z = \frac{1}{2}(1 \pm \sqrt{5})$$

Let q to be the Möbius transformation from $\left(\frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1-\sqrt{5})\right)$ to $(0, \infty)$

Then,

$$q(z) = \frac{z - \frac{1}{2}(1+\sqrt{5})}{z - \frac{1}{2}(1-\sqrt{5})}$$

# Continue

Note that, $q^{-1}(1) = \infty$ and $m(\infty) = 2$

then we have the multiplier of m

= $q \circ m \circ q^{-1}(1) = q(m(\infty)) = q(2) = \frac{3-\sqrt{5}}{3+\sqrt{5}}$

# Example

Let m be a Möbius transformation with two fixed points x and y. Prove that if $n_1$ and $n_2$ are two Möbius transformations satisfying $n_1(x) = 0 = n_2(x)$ and $n_1(y) = \infty = n_2(y)$, then the multipliers of $n_1 \circ m \circ n_1^{-1} = n_2 \circ m \circ n_2^{-1}$

## Answer

Let $n_1 \circ m \circ n_1^{-1}(z) = az$ and $n_2 \circ m \circ n_2^{-1}(z) = bz$

Since we have $n_2^{-1}(n_1(x)) = 0$ and $n_2^{-1}(n_1(y)) = \infty$

Let $p(z) = n_2(n_1^{-1}(z)) = cz$ for some c in $\mathbb{C}$ but not equal to 0 or 1

$bz = n_2 \circ m \circ n_2^{-1}(z)$

$= p \circ n_1 \circ m \circ n_1^{-1} \circ p^{-1}(z)$

$= p(\frac{a}{c}z) = az$

# Example

Using the notation of the argument just given for Möbius transformations with two fixed points, prove that if we conjugate m as above by a Möbius transformation s satisfying $s(x) = \infty$ and $s(y) = 0$ , the multiplier of $s \circ m \circ s^{-1} = \frac{1}{a}$

Answer

Let s be a Möbius transformation taking $(x, y)$ to $(\infty, 0)$
and q be a Möbius transformation taking $(x, y)$ to $(0, \infty)$

Let $q \circ m \circ q^{-1}(z) = az$

$$\text{Note that, } J(z) = \frac{1}{z} \text{ then } s = J \circ q$$
$$s \circ m \circ s^{-1}(z) = J \circ q \circ m \circ q^{-1} \circ J = \frac{1}{a} z$$

# 4. Reflection

# Complex Conjugation

Consider the simplest homeomorphism of $\overline{\mathbb{C}}$ not already in Möb$^+$: complex conjugation.

The function $C : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$C(z) = \overline{z} \text{ for } z \in \mathbb{C} \text{ and } C(\infty) = \infty$$

is an element of Homeo($\overline{\mathbb{C}}$).

Proof

Note that $C(\overline{z}) = z \text{ and } C(\infty) = \infty$,

Hence $C^{-1}(z) = C(z)$.

Hence $C$ is a bijection of $\overline{\mathbb{C}}$.

Let $z \in \overline{\mathbb{C}}$.

For any $\varepsilon > 0$, $C(U_\varepsilon(z)) = U_\varepsilon(C(z))$.

Hence $C$ is continuous.

Therefore, $C$ is is an element of Homeo($\overline{\mathbb{C}}$).

# Definition

## Möb

The general Möbius group Möb is the group generated by $\text{Möb}^+$ and C.

i.e. Every (nontrivial) element p of Möb can be expressed as a composition:

$$p = C \circ m_k \circ \cdots C \circ m_1$$

for some k ≥ 1, where each $m_k$ is an element of $\text{Möb}^+$.

# Theorem

$$\text{Möb} \subset \text{Homeo}^{\text{C}}(\overline{\mathbb{C}}).$$

## Proof

We have already proved that the elements of Möb$^+$ lie in $\text{Homeo}^{\text{C}}(\overline{\mathbb{C}})$ before.

Thus, we only have to prove that $C : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ lies in $\text{Homeo}^{\text{C}}(\overline{\mathbb{C}})$ to complete the proof.

Let A be a circle in $\overline{\mathbb{C}}$.

Suppose A is given by the equation $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$.

Set $w = C(z) = \overline{z}$.

Then $z = \overline{w}$.

Hence $w$ satisfies the equation $\alpha w\overline{w} + \overline{\beta} w + \beta\overline{w} + \gamma = 0$, which is a circle in $\overline{\mathbb{C}}$.

Then, $C : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ lies in $\text{Homeo}^{\text{C}}(\overline{\mathbb{C}})$.

Therefore, $\text{Möb} \subset \text{Homeo}^{\text{C}}(\overline{\mathbb{C}})$.

# Theorem

Every element of Möb has either the form: $m(z) = \dfrac{az+b}{cz+d}$ or $n(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ ,

where $a,\ b,\ c,\ d \in \mathbb{C}$ and $ad - bc \neq 0$ .

Proof

Note that the composition of two Möbius transformations is again a Möbius transformation.

Let $m(z) = \frac{az+b}{cz+d}$ , $n(z) = \frac{\alpha\bar{z}+\beta}{\gamma\bar{z}+\delta}$ and $p(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ ,

Then $(m \circ C)(z) = m(\bar{z}) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ , $(m \circ n)(z) = \dfrac{(a\alpha + b\gamma)\bar{z} + a\beta + b\delta}{(c\alpha + d\gamma)\bar{z} + c\beta + d\delta}$ and $(p \circ n)(z) = \dfrac{(a\bar{\alpha} + b\bar{\gamma})z + a\bar{\beta} + b\bar{\delta}}{(c\bar{\alpha} + d\bar{\gamma})z + c\bar{\beta} + d\bar{\delta}}$.

Therefore, it has the desired form for all cases.

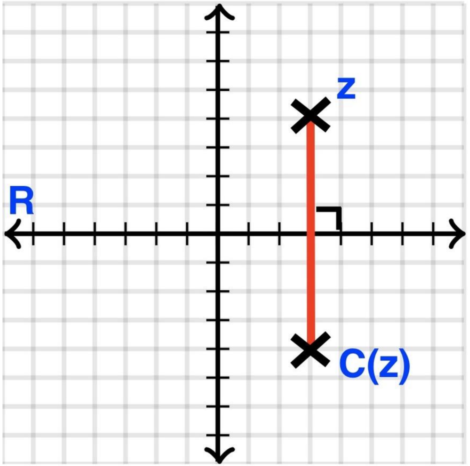
# Reflection

Geometrically, the action of $C$ on $\bar{\mathbb{C}}$ is reflection in the extended real axis $\overline{\mathbb{R}}$.

Given we have defined reflection in $\overline{\mathbb{R}}$, and given Möb acts transitively on the set $\mathcal{C}$ of circles in $\bar{\mathbb{C}}$, we are able to define reflection in any circle in $\bar{\mathbb{C}}$.

Particularly, let A be a circle in $\bar{\mathbb{C}}$, we can choose an element $m$ of Möb taking to A, and define reflection in A to be the composition:

$$C_A = m \circ C \circ m^{-1}$$



# Example

Let $A = \mathbb{S}^1$,

Let $m(z)$ be an element of Möb taking $\overline{\mathbb{R}}$ to S which is the transformation taking the triple (0, 1, ∞) to the triple (i, 1, -i),

Take $m(z) = \dfrac{\frac{1}{\sqrt{2}}z + \frac{i}{\sqrt{2}}}{\frac{i}{\sqrt{2}}z + \frac{1}{\sqrt{2}}}$.

Calculating, $C_A(z) = m \circ C \circ m^{-1}(z) = \dfrac{1}{\bar{z}} = \dfrac{z}{|z|^2}$.

# Proposition

Every element of Möb can be expressed as the composition of reflections in finitely many circles in $\overline{\mathbb{C}}$.

## Proof

Note that Möb is generated by Möb$^+$ and $C(z) = \overline{z}$, and as Möb$^+$ is generated by $J(z) = \frac{1}{z}$ and $f(z) = az + b$ for $a,\ b \in \mathbb{C}$ with $a \neq 0$, we only have to verify the proposition for $C(z)$ , $J(z)$ and $f(z)$.

For $C(z)$, $C(z)$ is a reflection by definition.

For $J(z)$, $J(z)$ can be expressed by the composition $C(z) = \overline{z}$ and the reflection $c(z) = \frac{1}{\overline{z}}$ in $\mathbb{S}^1$.

For $f(z)$ , $f(z)$ is the composition of $L(z) = az$ and $P(z) = z + b$, so what we left is to verify the proposition for $L(z)$ and $P(z)$ .

# Continue

For $P(z) = z + b$, let $b = \beta e^{i\varphi}$, let $\ell$ be the Euclidean line passing through 0 and b,
We express translation along $\ell$ as the reflection in two lines A and B perpendicular to $\ell$ ,
with A passing through 0 and B passing through $\frac{1}{2}b$.

Set $\theta = \varphi - \frac{1}{2}\pi$. Then we have:

$$C_A(z) = e^{2i\theta}\overline{z} = -e^{2i\varphi}\overline{z} \quad \text{and} \quad C_B(z) = -e^{2i\varphi}\left(\overline{z} - \frac{1}{2}\overline{b}\right) + \frac{1}{2}b$$

Therefore, $(C_B \circ C_A)(z) = C_B(-e^{2i\varphi}\overline{z}) = -e^{2i\varphi}\left(-e^{-2i\varphi}z - \frac{1}{2}\overline{b}\right) + \frac{1}{2}b = z + b.$

For $L(z) = az$, let $a = \alpha^2 e^{2i\theta}$, then $L(z)$ is the composition of $D(z) = \alpha^2 z$ and $E(z) = e^{2i\theta}z$.

For $D(z)$, $D(z)$ can be expressed by the composition of the reflection $c(z) = \frac{1}{\overline{z}}$ in $\mathbb{S}^1$ and
the reflection $c_2(z) = \frac{\alpha^2}{\overline{z}}$ in the Euclidean circle with Euclidean centre 0 and Euclidean radius $\alpha$ .

For $E(z)$ , $E(z)$ can be expressed by the composition of the reflection $C(z) = \overline{z}$ in $\mathbb{R}$ and
the reflection $C_2(z) = e^{i\theta}\overline{z}$ in the Euclidean line through 0 making angle $\theta$ with $\mathbb{R}$ .

Combining the result of the above cases, we have: every element of Möb can be expressed as the composition of reflections in finitely many circles in $\overline{\mathbb{C}}$ .

# Theorem

$\mathrm{M\ddot{o}b} = \mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}}).$

## Proof

We have proved that $\mathrm{M\ddot{o}b} \subset \mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}})$. What we left is to prove $\mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}}) \subset \mathrm{M\ddot{o}b}$.
Let $f \in \mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}})$, let $p$ be the Möbius transformation taking $(f(0), f(1), f(\infty))$ to $(0, 1, \infty)$.
Then, $p \circ f(0) = 0,\ p \circ f(1) = 1,$ and $p \circ f(\infty) = \infty.$
Note that $p \circ f$ takes circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$.
Since $p \circ f(\infty) = \infty$ and $\overline{\mathbb{R}}$ is the circle in $\overline{\mathbb{C}}$ determined by $(0, 1, \infty)$, we have $p \circ f(\mathbb{R}) = \mathbb{R}$.
Since $p \circ f$ takes $\mathbb{R}$ to $\mathbb{R}$ and fixes $\infty$, we have either $p \circ f(\mathbb{H}) = \mathbb{H}$ or is the lower half-plane.
For $p \circ f(\mathbb{H}) = \mathbb{H}$, we set $m = p.$
For the case of the lower half-plane, we set $m = C \circ p$ with $C$ being the complex conjugation.
Then, $m \circ f(0) = 0,\ m \circ f(1) = 1, m \circ f(\infty) = \infty,$ and $m \circ f(\mathbb{H}) = \mathbb{H}.$
Now, we want to prove that $m \circ f$ is the identity.
We will prove this by constructing dense set of points in $\overline{\mathbb{C}}$ such that each of which is fixed by $m \circ f$.

# Continue

Set $Z = \{z \in \overline{\mathbb{C}} \,|\, m \circ f(z) = z\}$.
Then, 0, 1 and $\infty$ are elements of $Z$ .
Since $m \circ f$ fixes $\infty$ and lies in $\mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}})(0) = 0,\ m \circ f(1) = 1,$
we have $m \circ f$ takes Euclidean lines in $\overline{\mathbb{C}}$ to Euclidean lines in $\overline{\mathbb{C}}$ ,
and $m \circ f$ takes Euclidean circles in $\overline{\mathbb{C}}$ to Euclidean circles in $\overline{\mathbb{C}}$ .
Suppose $X$ and $Y$ are two Euclidean lines in $\overline{\mathbb{C}}$ that intersect at some point $z_0$ ,
Further suppose that $m \circ f(X) = X$ and $m \circ f(Y) = Y$ ,
Then, $m \circ f(z_0) = z_0$ .
Hence, $z_0 \in m \circ f$ .
Let $s \in \mathbb{R}$,
Let $V(s)$ be the vertical line in $\overline{\mathbb{C}}$ through $s$ .
Let $H(s)$ be a horizontal line in $\overline{\mathbb{C}}$ through $is$ , where $i$ is the imaginary unit.
When s $\neq$ 0, since $H(s)$ and $\mathbb{R}$ are disjoint and $m \circ f(\mathbb{R}) = \mathbb{R}$,
we have $m \circ f(H)$ and $m \circ f(\mathbb{R}) = \mathbb{R}$ are disjoint.
Hence $H(s)$ is against a horizontal line in $\overline{\mathbb{C}}$.
Since $m \circ f(\mathbb{H}) = \mathbb{H}$ ,
we have $H(s)$ lies in $\mathbb{H}$ if and only if $m \circ f(H)$ lies in $\mathbb{H}$ .

# Continue

Let $A$ be the Euclidean circle with Euclidean centre $\frac{1}{2}$ and Euclidean radius $\frac{1}{2}$.
Then, $V(0)$ is tangent to $A$ at 0 and $V(1)$ is tangent to $A$ at 1.
Hence, $m \circ f(V(0))$ and $m \circ f(V(1))$ are the tangent lines to $m \circ f(A)$ at 0 and 1 respectively
Since $V(0)$ and $V(1)$ are parallel Euclidean lines in $\overline{\mathbb{C}}$,
we have $m \circ f(V(0))$ and $m \circ f(V(1))$ are parallel Euclidean lines in $\overline{\mathbb{C}}$ .
Hence, $m \circ f(V(0)) = V(0)$ and $m \circ f(V(1)) = V(1)$.
Since the tangent lines through 0 and 1 to any other Euclidean circle passing through 0 and 1 are not parallel,
we have $m \circ f(A) = A$.
Here, we want to find more points of $Z$ that $A$ contains other than 0 and 1.
Consider $H(\frac{1}{2})$ and $H(-\frac{1}{2})$, both of them are horizontal lines in $\overline{\mathbb{C}}$ ,
We can see that $H(\frac{1}{2})$ is tangent to $A$ at $\frac{1}{2}+\frac{1}{2}i$ and $H(-\frac{1}{2})$ is tangent to A at $-\frac{1}{2}+\frac{1}{2}i$ .
Since both of them are horizontal lines that tangent to $m \circ f(A) = A$,
we have $m \circ f(H(\frac{1}{2})) = H(\frac{1}{2})$ and $m \circ f(H(-\frac{1}{2})) = H(-\frac{1}{2})$.
Thus, we have more points in $Z$ , including:
$H(\frac{1}{2}) \cap V(0) = \frac{1}{2}i$ , $H(\frac{1}{2}) \cap V(1) = 1+\frac{1}{2}i$, $H(-\frac{1}{2}) \cap V(0) = -\frac{1}{2}i$ and $H(-\frac{1}{2}) \cap V(1) = 1-\frac{1}{2}i$

# Continue

Hence, Each pair of points in $Z$ gives rise to a Euclidean line that is taken to itself by $m \circ f$ .

Then, Each triple of noncollinear points in $Z$ gives rise to a Euclidean circle that is taken to itself by $m \circ f$ .

Thus, more points of $Z$ are found.

Hence, more Euclidean lines and Euclidean circles taken to themselves.

Then, $Z$ contains a dense set of points of $\overline{\mathbb{C}}$ .

Thus, $m \circ f$ is the identity.

Hence, $f = m^{-1}$ is an element of Möb.

Therefore, $\mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}}) \subset \mathrm{Möb}$ .

Combining with $\mathrm{Möb} \subset \mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}})$ , we have $\mathrm{Möb} = \mathrm{Homeo}^{\mathrm{C}}(\overline{\mathbb{C}})$.

# 5. Conformality of elements

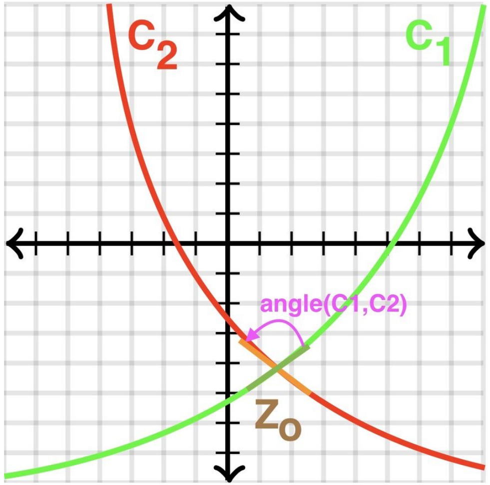
# Definition

## angle($C_1$, $C_2$)

Let $C_1$ and $C_2$ be two smooth curves in C that intersect at a point $z_0$ .

Define the angle angle($C_1$, $C_2$) between $C_1$and $C_2$ at $z_0$

to be the angle between the tangent lines to $C_1$ and $C_2$ at $z_0$ , measured from $C_1$ to $C_2$.




We adopt the following convention:
counterclockwise angles are positive and clockwise angles are negative.

Hence, angle($C_1$, $C_2$) = - angle($C_2$, $C_1$)

# Definition

## Conformality

A homeomorphism of $\overline{\mathbb{C}}$ that preserves the absolute value of the angle between curves is said to be conformal.

One major fact is that the elements of $\text{Möb}$ are conformal.

# Theorem

The elements of Möb are conformal homeomorphisms of $\bar{\mathbb{C}}$ .

## Proof

let $X_1$ and $X_2$ be two Euclidean lines in $\bar{\mathbb{C}}$ that intersect at a point $z_0$ ,
let $z_k$ be a point on $X_k$ such that $z_k \neq z_0$, and let $s_k$ be the slope of $X_k$ .

Hence, $s_k = \dfrac{\mathrm{Im}(z_k - z_0)}{\mathrm{Re}(z_k - z_0)}$.

Let $\theta_k$ be the angle that $X_k$ makes with the real axis $\mathbb{R}$,

Then, $s_k = \tan(\theta_k)$ and $\mathrm{angle}(X_1, X_2) = \theta_2 - \theta_1 = \arctan(s_2) - \arctan(s_1)$.

Note that Möb is generated by $\text{Möb}^+$ and $C(z) = \bar{z}$, and as $\text{Möb}^+$ is generated by $J(z) = \frac{1}{z}$ and $f(z) = az + b$ for $a,\ b \in \mathbb{C}$ with $a \neq 0$, we only have to verify the proposition for $C(z)$ , $J(z)$ and $f(z)$.

# Continue

For $f(z) = az + b$, write $a = \rho e^{i\beta}$.
Since $f(\infty) = \infty$, we have $f(X_1)$ and $f(X_2)$ are against Euclidean lines in $\mathbb{C}$.
Note that $f(X_k)$ passes through the points $f(z_0)$ and $f(z_k)$.
Let $t_k$ be the slope of $f(X_k)$ ,

$$t_k = \frac{\mathrm{Im}(f(z_k) - f(z_0))}{\mathrm{Re}(f(z_k) - f(z_0))} = \frac{\mathrm{Im}(a(z_k - z_0))}{\mathrm{Re}(a(z_k - z_0))}$$

$$= \frac{\mathrm{Im}(e^{i\beta}(z_k - z_0))}{\mathrm{Re}(e^{i\beta}(z_k - z_0))} = \tan(\beta + \theta_k).$$

Hence, $\mathrm{angle}(f(X_1), f(X_2)) = \arctan(t_2) - \arctan(t_1)$

$$= (\beta + \theta_2) - (\beta + \theta_1)$$

$$= \theta_2 - \theta_1 = \mathrm{angle}(X_1, X_2)$$

Therefore, $f(z)$ is conformal.

# Continue

For $J(z) = \frac{1}{z}$, $J(X_1)$ and $J(X_2)$ may not only be two Euclidean lines in $\mathbb{C}$.
They may be both Euclidean circles in $\mathbb{C}$, or may be one Euclidean line and one Euclidean circle.
Here, we prove for the case that both of them are Euclidean circles in $\mathbb{C}$.
We may suppose that $X_k$ is given as the set of solutions to the following equation:

$$\beta_k z + \overline{\beta_k}\overline{z} + 1 = 0 \quad \text{where } \beta_k \in \mathbb{C}$$

Let $s_k$ be the slope of $X_k$, then $s_k = \dfrac{\mathrm{Re}(\beta_k)}{\mathrm{Im}(\beta_k)}$.

Hence, $J(X_k)$ can be given as the set of solutions to the following equation:

$$z\overline{z} + \overline{\beta_k} z + \beta_k \overline{z} = 0$$ , which is equivalent to $|z + \beta_k|^2 = |\beta_k|^2$

Thus, the slope of the tangent line to $J(X_k)$ at 0 is $-\dfrac{\mathrm{Re}(\beta_k)}{\mathrm{Im}(\beta_k)} = -\tan(\theta_k) = \tan(-\theta_k)$

Then, $\mathrm{angle}(J(X_1), J(X_2)) = -\theta_2 - (-\theta_1) = -\mathrm{angle}(X_1, X_2)$

Hence, the absolute value of the angle between curves is preserved.
Therefore, $J(z)$ is conformal.

# Continue

For $C(z) = \overline{z}$ , since $X_k$ passes through $z_k$ and $z_0$,
we have $C(X_k)$ passes through $C(z_0) = \overline{z_0}$ and $C(z_k) = \overline{z_k}$.

Let $S_k$ be the slope of $C(X_k)$, then $S_k = \frac{\mathrm{Im}(\overline{z_k} - \overline{z_0})}{\mathrm{Re}(\overline{z_k} - \overline{z_0})} = -\frac{\mathrm{Im}(z_k - z_0)}{\mathrm{Re}(z_k - z_0)} = -s_k$.

Hence, $\begin{aligned} \mathrm{angle}(C(X_1), C(X_2)) &= \arctan(S_2) - \arctan(S_1) \\ &= -\arctan(s_2) + \arctan(s_1) = -\mathrm{angle}(X_1, X_2). \end{aligned}$

Hence, the absolute value of the angle between curves is preserved.
Therefore, $C(z)$ is conformal.

Combining the result of the three above cases, we have:
The elements of Möb are conformal homeomorphisms of $\overline{\mathbb{C}}$ .

# 6. Preserving H and transitivity properties

# Definition

In order to find transformations that take hyperbolic lines in $\mathbb{H}$ to hyperbolic lines in $\mathbb{H}$ ,

Let's consider the following group:

$$\text{Möb}(\mathbb{H}) = \{m \in \text{Möb} \mid m(\mathbb{H}) = \mathbb{H}\}.$$

# Theorem

Every element of $\text{Möb}(\mathbb{H})$ takes hyperbolic lines in $\mathbb{H}$ to hyperbolic lines in $\mathbb{H}$.

## Proof

The proof of this theorem is the immediate consequence of the previous theorem:

The elements of Möb are conformal homeomorphisms of $\bar{\mathbb{C}}$.

That is:

1) The elements of $\text{Möb}(\mathbb{H})$ preserve angles between circles in $\bar{\mathbb{C}}$
2) Every hyperbolic line in $\mathbb{H}$ is the intersection of $\mathbb{H}$ with a circle in $\bar{\mathbb{C}}$ perpendicular to $\bar{\mathbb{R}}$
3) Every element of Möb takes circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$.

# Definition

Let's consider the following groups:

$$\text{Möb}(\overline{\mathbb{R}}) = \{m \in \text{Möb} \mid m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}\}$$

$$\downarrow$$

$$\text{Möb}(\mathbb{H}) = \{m \in \text{Möb} \mid m(\mathbb{H}) = \mathbb{H}\}$$

$$\downarrow$$

$$\text{Möb}^+(\mathbb{H}) = \{m \in \text{Möb}^+ \mid m(\mathbb{H}) = \mathbb{H}\}$$

# Recall: Theorem

Every element of Möb has either the form: $m(z) = \dfrac{az+b}{cz+d}$ or $n(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$,

where $a,\ b,\ c,\ d \in \mathbb{C}$ and $ad - bc \neq 0$ .

Proof

Note that the composition of two Möbius transformations is again a Möbius transformation.

Let $m(z) = \frac{az+b}{cz+d}$, $n(z) = \frac{\alpha\bar{z}+\beta}{\gamma\bar{z}+\delta}$ and $p(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$,

Then $(m \circ C)(z) = m(\bar{z}) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$, $(m \circ n)(z) = \dfrac{(a\alpha + b\gamma)\bar{z} + a\beta + b\delta}{(c\alpha + d\gamma)\bar{z} + c\beta + d\delta}$ and $(p \circ n)(z) = \dfrac{(a\bar{\alpha} + b\bar{\gamma})z + a\bar{\beta} + b\bar{\delta}}{(c\bar{\alpha} + d\bar{\gamma})z + c\bar{\beta} + d\bar{\delta}}$.

Therefore, it has the desired form for all cases.

# Continue

Every element of Möb has either the form: $m(z) = \frac{az+b}{cz+d}$ or $n(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$,

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ .

Since $C(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$, we have $m \circ C(z) = m(\bar{z}) = \frac{az+b}{cz+d}$.

Hence, we only have to consider $m(z) = \frac{az+b}{cz+d}$ and limit the constraint to $ad - bc = 1$ .

Then, $m^{-1}(\infty) = -\frac{d}{c}$, $m(\infty) = \frac{a}{c}$, and $m^{-1}(0) = -\frac{b}{a}$ all lie in $\overline{\mathbb{R}}$.

# Continue

Case 1: Suppose $a \neq 0$ and $c \neq 0$,

Hence $a = m(\infty)c$, $b = -m^{-1}(0)a = -m^{-1}(0)m(\infty)c$, and $d = -m^{-1}(\infty)c$.

Then $m(z) = \dfrac{az+b}{cz+d} = \dfrac{m(\infty)cz - m^{-1}(0)m(\infty)c}{cz - m^{-1}(\infty)c}$

Then,
$$\begin{aligned} 1 = ad - bc &= c^2\left[-m(\infty)m^{-1}(\infty) + m(\infty)m^{-1}(0)\right] \\ &= c^2\left[m(\infty)(m^{-1}(0) - m^{-1}(\infty))\right]. \end{aligned}$$

Since $m(\infty)$, $m^{-1}(0)$, and $m^{-1}(\infty)$ are all real,

we have $c$ is either real or purely imaginary.

Hence, $a$, $b$, $c$, and $d$ are either all real or all purely imaginary.

# Continue

Case 2: Suppose $a = 0$,

Hence $c \neq 0$ and then $m(1) = \frac{b}{c+d}$ and $m^{-1}(\infty) = -\frac{d}{c}$.

Then, $d = -m^{-1}(\infty)c$ and $b = m(1)(c+d) = (m(1) - m^{-1}(\infty))c$.

Then, $1 = ad - bc = (m^{-1}(\infty) - m(1))c^2$.

Then, $c$ is either real or purely imaginary.

Hence, $a,\ b,\ c,$ and $d$ are either all real or all purely imaginary.

# Continue

Case 3: Suppose $c = 0$ ,

Hence $a \neq 0$ and $d \neq 0$ and then both $m(0) = \frac{b}{d}$ and $m(1) = \frac{a+b}{d}$ are real.

Then, $b = m(0)d$ and $a = (m(1) - m(0))d.$

Then, $1 = ad - bc = (m(1) - m(0))d^2$ .

Then, $d$ is either real or purely imaginary.

Hence, $a,\ b,\ c,$ and $d$ are either all real or all purely imaginary.

# Continue

Conversely, suppose $m$ has either the form $m(z) = \frac{az+b}{cz+d}$ or $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ with $ad - bc = 1$

Further suppose $a,\ b,\ c,\ \text{and}\ d$ are either all real or all purely imaginary,

Then $m(\infty),\ m^{-1}(0),\ \text{and}\ m^{-1}(\infty)$ are all lie on $\overline{\mathbb{R}}$,

Therefore, $m$ takes $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ .

# Theorem

Every element of $\text{Möb}(\overline{\mathbb{R}})$ has one of the following four forms:

1. $m(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$

2. $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$

3. $m(z) = \frac{az+b}{cz+d}$ with $a, b, c, d$ purely imaginary and $ad - bc = 1$

4. $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d$ purely imaginary and $ad - bc = 1$

# Continue

Case 1: Suppose $m(z) = \frac{az+b}{cz+d}$, where $a$, $b$, $c$, and $d$ are real such that $ad - bc = 1$.

Hence, 
$$\begin{aligned} \operatorname{Im}(m(i)) &= \operatorname{Im}\left(\frac{ai+b}{ci+d}\right) \\ &= \operatorname{Im}\left(\frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)}\right) = \frac{ad-bc}{c^2+d^2} = \frac{1}{c^2+d^2} > 0, \end{aligned}$$

Therefore, $m \in \text{Möb}(\mathbb{H})$ in this case.

# Continue

Case 2: Suppose $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $a,\ b,\ c,\ \text{and}\ d$ are real such that $ad - bc = 1$.

Hence, $$\begin{aligned} \operatorname{Im}(m(i)) &= \operatorname{Im}\left(\frac{-ai+b}{-ci+d}\right) \\ &= \operatorname{Im}\left(\frac{(-ai+b)(ci+d)}{(-ci+d)(ci+d)}\right) = \frac{-ad+bc}{c^2+d^2} = \frac{-1}{c^2+d^2} < 0, \end{aligned}$$

Therefore, $m \notin \text{Möb}(\mathbb{H})$ in this case.

# Continue

Case 3: Suppose $m(z) = \frac{az+b}{cz+d}$, where $a,\ b,\ c,$ and $d$ are purely imaginary such that $ad - bc = 1$.

Write $a = \alpha i,\ b = \beta i,\ c = \gamma i,$ and $d = \delta i$ such that $\alpha\delta - \beta\gamma = -1.$

Hence,
$$\begin{aligned}\mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{ai+b}{ci+d}\right) = \mathrm{Im}\left(\frac{-\alpha+\beta i}{-\gamma+\delta i}\right)\\ &= \mathrm{Im}\left(\frac{(-\alpha+\beta i)(-\gamma-\delta i)}{(-\gamma+\delta i)(-\gamma-\delta i)}\right) = \frac{\alpha\delta-\beta\gamma}{\gamma^2+\delta^2} = \frac{-1}{\gamma^2+\delta^2} < 0,\end{aligned}$$

Therefore, $m \notin \mathrm{M\ddot{o}b}(\mathbb{H})$ in this case.

# Continue

Case 4: Suppose $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $a,\ b,\ c,\ \text{and}\ d$ are purely imaginary such that $ad - bc = 1$.

Write $a = \alpha i,\ b = \beta i,\ c = \gamma i,\ \text{and}\ d = \delta i$ such that $\alpha\delta - \beta\gamma = -1.$

Hence, $$\begin{aligned} \operatorname{Im}(m(i)) &= \operatorname{Im}\left(\frac{-ai+b}{-ci+d}\right) = \operatorname{Im}\left(\frac{\alpha+\beta i}{\gamma+\delta i}\right) \\ &= \operatorname{Im}\left(\frac{(\alpha+\beta i)(\gamma-\delta i)}{(\gamma+\delta i)(\gamma-\delta i)}\right) = \frac{-\alpha\delta+\beta\gamma}{\gamma^2+\delta^2} = \frac{1}{\gamma^2+\delta^2} > 0, \end{aligned}$$

Therefore, $m \in \text{Möb}(\mathbb{H})$ in this case.

# Theorem

Every element of $\text{Möb}(\mathbb{H})$ has one of the following two forms:

1. $m(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$
2. $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, where $a, b, c, d$ are purely imaginary and $ad - bc = 1$

# Continue

No element of $\text{Möb}(\mathbb{H})$ of the form:

$$n(z) = \frac{a\bar{z} + b}{c\bar{z} + d}, \text{ where } a,\ b,\ c,\ d \text{ are purely imaginary and } ad - bc = 1$$

can be an element of $\text{Möb}^+(\mathbb{H})$ .

# Theorem

Every element of $\text{Möb}^+(\mathbb{H})$ has following form:

1. $m(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$

# Proposition

Reflection in a circle in $\overline{\mathbb{C}}$ is well defined.

## Proof

Let $m \in \mathrm{Möb}(\mathbb{R})$ , let $C(z) = \overline{z}$.

For $m(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ ,

$$C \circ m(z) = \frac{a\overline{z} + b}{c\overline{z} + d} = m \circ C(z)$$

For $m(z) = \frac{az+b}{cz+d}$ with $a, b, c, d$ purely imaginary and $ad - bc = 1$ ,

$$C \circ m(z) = \frac{-az - b}{-cz - d} = \frac{az + b}{cz + d} = m \circ C(z)$$

Let $A$ be a circle in $\overline{\mathbb{C}}$, let $m, n \in \mathrm{Möb}(\overline{\mathbb{R}})$, both taking $\overline{\mathbb{R}}$ to $A$.
Then, $n^{-1} \circ m$ takes $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ , thus $n^{-1} \circ m = p$ for some element $p$ of $\mathrm{Möb}(\overline{\mathbb{R}})$.
In particular, $p \circ C = C \circ p$. Write $m = n \circ p$.

$$m \circ C \circ m^{-1} = n \circ p \circ C \circ p^{-1} \circ n^{-1} = n \circ p \circ p^{-1} \circ C \circ n^{-1} = n \circ C \circ n^{-1}$$

Therefore, Reflection in a circle in $\overline{\mathbb{C}}$ is well defined.

# Recall: Lemma

A group *G* acts on a set *X* if there is a homomorphism from *G* in to the group bij(*X*) of bijections of *X*

### Definition

*G* acts transitively on *X* if for each pair *x* and *y* of elements of *X*, there exist some element *g* of *G* satisfying *g*(*x*) = *y*

## Lemma

Suppose a group *G* acts on a set *X*, and let $x_0$ be a point of *X*. Suppose for each point *y* of *X*, there exists an element *g* of G so that $g(y) = x_0$. Then, *G* acts transitively on a set *X*

# Proposition

$\text{Möb}(\mathbb{H})$ acts transitively on $\mathbb{H}$.

## Proof

Let $w \in \mathbb{H}$, it is sufficient to show that $\exists$ m $\in \text{Möb}(\mathbb{H})$ such that $m(w) = i$.

Let $w = a + bi$, where $a, b \in \mathbb{R}$ and $b > 0$.

Let $p(z) = z - a$, thus $p(w) = p(a + bi) = bi$.

Let $q(z) = \frac{1}{b}z$, thus $q(p(w)) = q(bi) = i$.

Note that $-a \in \mathbb{R}$ and $\frac{1}{b} > 0$.

Hence, $p(z) \in \text{Möb}(\mathbb{H})$ and $q(z) \in \text{Möb}(\mathbb{H})$ .

Then, $q \circ p(z) \in \text{Möb}(\mathbb{H})$.

Therefore, $\text{Möb}(\mathbb{H})$ acts transitively on $\mathbb{H}$.

# Definition

Let $\ell$ be a hyperbolic line in $\mathbb{H}$ .

Open half-plane in $\mathbb{H}$ : a component of the complement of $\ell$ .

Closed half-plane in $\mathbb{H}$ : the union of $\ell$ with one of the open half- planes determined by $\ell$ .

Half-plane in $\mathbb{H}$ : either open half-plane or closed half-plane in $\mathbb{H}$ .

# Recall: Lemma

A group $G$ acts on a set $X$ if there is a homomorphism from $G$ in to the group bij($X$) of bijections of $X$

## Definition

$G$ acts transitively on $X$ if for each pair $x$ and $y$ of elements of $X$, there exist some element $g$ of $G$ satisfying $g(x) = y$

## Lemma

Suppose a group $G$ acts on a set $X$, and let $x_0$ be a point of $X$. Suppose for each point $y$ of $X$, there exists an element $g$ of G so that $g(y) = x_0$. Then, $G$ acts transitively on a set $X$

# Proposition

$\text{Möb}(\mathbb{H})$ acts triply transitively on the set $\mathcal{T}_{\overline{\mathbb{R}}}$ of triples of distinct points of $\overline{\mathbb{R}}$ .

## Proof

Let $(z_1, z_2, z_3)$ be a triple of distinct points of $\overline{\mathbb{R}}$ .

It is sufficient to prove that ∃ m∈ $\text{Möb}(\mathbb{H})$ such that m takes $(z_1, z_2, z_3)$ to $(0, 1, \infty)$ .

Let $\ell$ be the hyperbolic line whose endpoints at infinity are $z_1$ and $z_3$ ,

Let $m$ be an element of $\text{Möb}(\mathbb{H})$ taking $\ell$ to the positive imaginary axis $I$ .

Assume that $m(z_1) = 0$ and $m(z_3) = \infty$ as we can composing $m$ with $K(z) = -\frac{1}{z}$ if necessary.

Set $b = m(z_2)$.

If $b > 0$, then the composition of $m$ with $p(z) = \frac{1}{b}z$ takes $(z_1, z_2, z_3)$ to $(0, 1, \infty)$.

If $b < 0$, then the composition of $m$ with $q(z) = \frac{1}{b}\overline{z}$ takes $(z_1, z_2, z_3)$ to $(0, 1, \infty)$

Therefore, $\text{Möb}(\mathbb{H})$ acts triply transitively on the set $\mathcal{T}_{\overline{\mathbb{R}}}$ of triples of distinct points of $\overline{\mathbb{R}}$ .

# 7. Conclusion

# We have talked about the following topics:

1. Transitivity
2. Transfomration
3. Relfection
4. Conformality of elements
5. Preserving H and transitivity properties

# 8. Reference

# Reference


Hyperbolic geometry, by James W. Anderson, Springer, 1999.

https://math.stackexchange.com/questions/481631/circle-on-riemann-sphere/481645

https://en.wikipedia.org/wiki/Riemann_sphere

Thank you very much!